

$N=2$ AND $N=4$ SUBALGEBRAS OF SUPER VERTEX OPERATOR ALGEBRAS

GEOFFREY MASON, MICHAEL TUIE AND GAYWALEE YAMSKULNA

ABSTRACT. We develop criteria to decide if an $N=2$ or $N=4$ super conformal algebra is a subalgebra of a super vertex operator algebra in general, and of a super lattice theory in particular. We give some specific examples.

1. INTRODUCTION

The advent of Mathieu Moonshine [EOT] in recent years has brought renewed attention to the super conformal $N=4$ algebra $\mathcal{A}_{N=4}$. Song has shown [S] that for central charge $c=6$, $\mathcal{A}_{N=4}$ is the algebra of global sections of the chiral de Rham complex on a $K3$ surface (following earlier results for hyperkähler manifolds [BZHS]), clarifying the connection between $K3$ and $\mathcal{A}_{N=4}$ as long understood by physicists.

All of this means that one can expect $\mathcal{A}_{N=4}$ to play a ubiquitous rôle in the further investigation of these subjects, much as the Virasoro algebra does in general CFT. Now the precise definition of $\mathcal{A}_{N=4}$ is awkward, to say the least. For $c=6$, it is usually described as a subtheory of the algebra of 12 free fermions. When we found ourselves looking for $\mathcal{A}_{N=4}$ in a super lattice theory containing no free fermions [MTY], there was a dearth of results in the literature to which we could turn.

The purpose of the present paper is to alleviate this situation. We prove two general recognition theorems which allow one to identify $\mathcal{A}_{N=4}$ as a subalgebra of a suitable Super Vertex Operator Algebra (SVOA) with just a few well-chosen axioms. Experts will perhaps not be surprised by the results, however some effort is required to obtain efficient characterizations, i.e., without too many assumptions. It would be of interest if a genuine reduction in the number of axioms needed can be achieved in our recognition theorems. Our main results are as follows (unexplained notation is clarified below).

Theorem 1. *Let U be a SVOA of CFT-type. Let $V \subseteq U$ be the subalgebra generated by 4 primary vectors of weight $\frac{3}{2}$ in U , so that*

$$V = \mathbb{C}\mathbf{1} \oplus V_{\frac{1}{2}} \oplus V_1 \oplus V_{\frac{3}{2}} \oplus \dots$$

is a conformally graded subspace of U . Assume that the following hold:

- (I) *The subspace of $V_{\frac{3}{2}}$ spanned by the four generators decomposes as $A \oplus B$, a pair of 2-dimensional vector representations for \mathfrak{sl}_2 , where*
- (II) $A(1)B \cong \mathfrak{sl}_2$,
- (III) $A(0)A = B(0)B = 0$,
- (IV) $T\mathfrak{sl}_2 \cap A(0)B \neq 0$.

1991 *Mathematics Subject Classification.* Primary, Secondary .

Key words and phrases. $N=4$ algebra, super vertex operator algebra.

The first author was supported by the NSF and the Simons Foundation #427007. The third author was supported by the Simons Foundation Collaboration Grant # 207862.

Then $V \cong \mathcal{A}_{N=4}$ with central charge $c=6k$, where $k \in \mathbb{C}$ is the level of the $\widehat{\mathfrak{sl}}_2$ Kac-Moody subalgebra generated by $A(1)B$.

Theorem 2. Let L be a positive-definite, odd, integral lattice of minimum norm 2 with V_L the corresponding SVOA. Let $V \subseteq V_L$ be the subalgebra generated by 4 vectors of weight $\frac{3}{2}$, so that

$$V = \mathbb{C}\mathbf{1} \oplus V_1 \oplus V_{\frac{3}{2}} \oplus \dots,$$

and assume that the following hold:

- (I) The subspace of $V_{\frac{3}{2}}$ spanned by the four generators decomposes as $A \oplus B$, a pair of 2-dimensional vector representations for \mathfrak{sl}_2 , where
- (II) $A(1)B \cong \mathfrak{sl}_2$,
- (III) $A(1)A = B(1)B = 0$,
- (IV) \mathfrak{sl}_2 contains a root of L .

Then $V \cong \mathcal{A}_{N=4}$ with central charge $c=6$.

The paper is organized as follows. After reviewing some background about the $N=4$ algebra in Section 2, we give the proof of Theorems 1 and 2 in Sections 3 and 4 respectively. We also briefly describe the more elementary analogous case of the $N=2$ superconformal subalgebra. This is contained in Section 5. In Section 6 we illustrate how the main Theorems may be applied to some examples. In particular, we give (Proposition 26) a painless new construction of the $N=4$ algebra in a certain rank 6 lattice theory V_L containing no free fermions. We include some appendices in Section 7 containing technical background in SVOA theory that we assume and use throughout the paper.

Acknowledgements We thank J. Duncan, R. Heluani, M. Miyamoto and R. Volpato for their comments and corrections to an earlier version of this paper.

2. $N=4$ SUPERCONFORMAL ALGEBRAS

The abstract generators and relations for the $N=4$ super conformal algebra $\mathcal{A}_{N=4}$ of central charge c are as follows. It is generated by 4 states G^\pm, \overline{G}^\pm of conformal weight $\frac{3}{2}$. The nontrivial relations can be expressed as follows (e.g. [ET, K])

- | | | |
|---|---|--|
| (a) $J^0(0)J^\pm = \pm 2J^\pm,$ | (b) $J^0(1)J^0 = \frac{c}{3}\mathbf{1},$ | (c) $J^+(0)J^- = J^0,$ |
| (d) $J^+(1)J^- = \frac{c}{6}\mathbf{1},$ | (e) $J^0(0)G^\pm = \pm G^\pm$ | (f) $J^0(0)\overline{G}^\pm = \pm \overline{G}^\pm,$ |
| (g) $J^\pm(0)G^\mp = G^\pm,$ | (h) $J^\pm(0)\overline{G}^\mp = -\overline{G}^\pm,$ | (i) $G^\pm(1)\overline{G}^\pm = 2J^\pm,$ |
| (j) $G^\pm(1)\overline{G}^\mp = \pm J^0,$ | (k) $G^\pm(2)\overline{G}^\mp = \frac{c}{3}\mathbf{1},$ | (l) $G^\pm(0)\overline{G}^\pm = T J^\pm,$ |
| (m) $G^\pm(0)\overline{G}^\mp = \omega \pm \frac{1}{2}T J^0.$ | | |

Here, ω is a Virasoro element of central charge c , T is the translation operator and J^\pm, J^0 are weight one vectors. Relations (a)–(m) hold in addition to the usual Virasoro relations between J^\pm, J^0 and ω . The initial segment of the Fock space of $V = \mathcal{A}_{N=4}$ is

$$V = \mathbb{C}\mathbf{1} \oplus V_1 \oplus V_{\frac{3}{2}} \oplus V_2 \oplus \dots$$

where

$$\begin{aligned} V_1 &= \langle J^\pm, J^0 \rangle \cong \mathfrak{sl}_2 \\ V_{\frac{3}{2}} &= \langle G^\pm \rangle \oplus \langle \overline{G}^\pm \rangle, \text{ a pair of vector representations for } \mathfrak{sl}_2 \\ V_2 &= T \mathfrak{sl}_2 \oplus \mathbb{C}\omega. \end{aligned}$$

For arbitrary subsets $X, Y \subseteq U$ and any integer n , we have already used, and will continue to use, the convenient notation $X(n)Y$ for the linear subspace of U spanned by all products $u(n)v$ ($u \in X, v \in Y$). For the $N=4$ algebra $\mathcal{A}_{N=4}$ we set

$$A := \langle G^\pm \rangle, \quad B := \langle \overline{G}^\pm \rangle.$$

Then it follows from the relations (a)–(m) that

$$A(n)A = B(n)B = 0 \quad (n \geq 0), \quad A(0)B = V_2, \quad A(1)B = \mathfrak{sl}_2, \quad A(2)B = \mathbb{C}1.$$

3. PROOF OF THEOREM 1

Let ω^U be the Virasoro element of U , with vertex operator

$$Y(\omega^U, z) = \sum_{n \in \mathbb{Z}} L^U(n) z^{-n-2}.$$

3.1. Assumptions (I) and (II). In the interests of keeping track of just which of the axioms (I)–(IV) in the statement of Theorem 1 are needed when, we begin by recording some consequences of axioms (I) and (II) alone. We set

$$\begin{aligned} (1) \quad \mathfrak{sl}_2 &:= \langle h, x^\pm \rangle \subseteq V_1 \\ (2) \quad A &:= \langle \tau^\pm, \tau^\mp \rangle, \quad B := \langle \overline{\tau}^\pm, \overline{\tau}^\mp \rangle. \end{aligned}$$

Here,

$$(3) \quad h(0)x^\pm = \pm 2x^\pm, \quad x^+(0)x^- = h$$

are standard generators and relations for \mathfrak{sl}_2 , and $\tau^\pm, \overline{\tau}^\pm$ are weight vectors for $h(0)$ with weights ± 1 , with

$$\begin{aligned} (4) \quad x^\pm(0)\tau^\mp &= \tau^\pm, \quad x^\pm(0)\overline{\tau}^\mp = \overline{\tau}^\pm, \\ (5) \quad x^\pm(0)\tau^\pm &= x^\pm(0)\overline{\tau}^\pm = 0. \end{aligned}$$

All of this is just a choice of notation based on the hypotheses of the Theorem 1. It amounts to the existence of an isomorphism of \mathfrak{sl}_2 -modules

$$\varphi: A \rightarrow B, \quad \tau^\pm \mapsto \overline{\tau}^\pm.$$

Note that some latitude in scaling the generators of A and B remains - a fact that we make use of later. In any case, there is a canonical \mathfrak{sl}_2 -invariant decomposition into trivial and adjoint modules

$$A \otimes B = \Lambda \oplus \Sigma,$$

where

$$\begin{aligned} (6) \quad \Lambda &= \mathbb{C}(\tau^+ \otimes \overline{\tau}^- - \tau^- \otimes \overline{\tau}^+), \\ (7) \quad \Sigma &= \langle \tau^+ \otimes \overline{\tau}^+, \tau^- \otimes \overline{\tau}^-, \tau^+ \otimes \overline{\tau}^- + \tau^- \otimes \overline{\tau}^+ \rangle. \end{aligned}$$

Of course, there is a similar decomposition of $A \otimes A = \Lambda^2(A) \oplus S^2(A)$.

Note that $A(1)B$ and $A \oplus B$ consist of primary states. This is clear for $A \oplus B$, while for $A(1)B$ we note that $L^U(1):V_1 \rightarrow \mathbb{C}\mathbf{1}$ is a morphism of \mathfrak{sl}_2 -modules. Restriction to $A(1)B = \mathfrak{sl}_2$ must therefore be trivial, and the assertion follows. Thus we have

$$(8) \quad [L^U(m), u(n)] = -nu(m+n),$$

$$(9) \quad [L^U(m), v(n)] = \left(\frac{1}{2}(m+1) - n \right) v(m+n),$$

for all $u \in A(1)B$ and $v \in A \oplus B$. In particular, (9) implies

Lemma 3. $A(n+1)A = L^U(1)A(n)A$ for all $n \neq 1$ and $A(n+2)A = L^U(2)A(n)A$ for all n . Similar relations hold for $B(n+1)B$, $B(n+2)B$, $A(n+1)B$ and $A(n+2)B$. \square

For $u, v \in A(1)B = \mathfrak{sl}_2$ we have from commutativity that

$$(10) \quad [u(m), v(n)] = (u(0)v)(m+n) + m\langle\langle u, v \rangle\rangle \delta_{m+n,0},$$

with $\langle\langle u, v \rangle\rangle$ given by $u(1)v = \langle\langle u, v \rangle\rangle \mathbf{1}$. Define $k \in \mathbb{C}$ by

$$(11) \quad \langle\langle h, h \rangle\rangle = 2k.$$

Lemma 4. V is a module for the Kac-Moody algebra $\widehat{\mathfrak{sl}}_2$ of level k .

Proof. Since $h(1)x^\pm \in V_0 = \mathbb{C}\mathbf{1}$ we find

$$0 = h(0)(h(1)x^\pm) = h(1)h(0)x^\pm = \pm 2h(1)x^\pm,$$

so that $\langle\langle h, x^\pm \rangle\rangle = 0$. Using $x^+(0)x^- = h$ this implies

$$(12) \quad 2k\mathbf{1} = h(1)(x^+(0)x^-) = 2x^+(1)x^-,$$

by commutativity. Hence, for $k \neq 0$, $\langle\langle \cdot, \cdot \rangle\rangle$ is a nondegenerate, invariant, bilinear form on \mathfrak{sl}_2 , so that for all $u, v \in \mathfrak{sl}_2$

$$(13) \quad \langle\langle u, v \rangle\rangle = k(u, v),$$

where here (\cdot, \cdot) is the standard invariant bilinear form normalized to $(h, h) = 2$. (13) also holds for $k = 0$. Hence, since $u(0)v = -v(0)u$ defines a commutator bracket on \mathfrak{sl}_2 , (10) implies the desired result. \square

Lemma 5. Properties (a)–(h) above hold for $c = 6k$.

Proof. We set

$$(14) \quad J^0 := h, \quad J^\pm := x^\pm, \quad G^\pm := \tau^\pm, \quad \text{and} \quad \overline{G}^\pm := \mp \overline{\tau}^\pm.$$

By (3)–(5), (11)–(12), we have

$$\begin{aligned} J^0(0)J^\pm &= \pm 2J^\pm, & J^0(1)J^0 &= 2k\mathbf{1}, \\ J^+(0)J^- &= J^0, & J^+(1)J^- &= k\mathbf{1}, \\ J^0(0)G^\pm &= \pm G^\pm, & J^0(0)\overline{G}^\pm &= \pm \overline{G}^\pm, \\ J^\pm(0)G^\mp &= G^\pm, & J^\pm(0)\overline{G}^\mp &= -\overline{G}^\pm. \end{aligned}$$

These are the needed relations. \square

Remark 6. In addition, we have $J^\pm(0)G^\pm = J^\pm(0)\overline{G}^\pm = 0$.

Next, we will obtain properties (i) and (j):

Lemma 7. *We can choose the normalization for $\tau^\pm, \bar{\tau}^\pm$ so that*

$$(15) \quad h = \tau^\pm(1)\bar{\tau}^\mp, \quad \tau^\pm(1)\bar{\tau}^\pm = \mp 2x^\pm.$$

In particular, we have $G^\pm(1)\bar{G}^\pm = 2J^\pm$, and $G^\pm(1)\bar{G}^\mp = \pm J^0$.

Proof. Thanks to the \mathfrak{sl}_2 -morphism $u \otimes v \mapsto u(1)v$ from $A \otimes B$ on to $A(1)B$, we get generators of $A(1)B$ just by replacing $u \otimes v$ by $u(1)v$ in the generators for Σ in (7). Thus

$$A(1)B = \mathfrak{sl}_2 = \langle \tau^+(1)\bar{\tau}^+, \tau^-(1)\bar{\tau}^-, \tau^+(1)\bar{\tau}^- + \tau^-(1)\bar{\tau}^+ \rangle.$$

Similarly, the generator for Λ in (6) maps to 0, leading to $\tau^+(1)\bar{\tau}^- - \tau^-(1)\bar{\tau}^+ = 0$.

$\tau^+(1)\bar{\tau}^- + \tau^-(1)\bar{\tau}^+ = 2\tau^\pm(1)\bar{\tau}^\mp$ is a nonzero semisimple element of \mathfrak{sl}_2 . As such, it is a nonzero multiple of h . Since we are free to simultaneously scale the τ 's by a nonzero constant, we may choose the scale so that $h = \tau^+(1)\bar{\tau}^- = \tau^-(1)\bar{\tau}^+$. Since $x^\pm(0)h = -h(0)x^\pm = \mp 2x^\pm$ we find, using super commutativity, that

$$\begin{aligned} -2x^+ &= x^+(0) (\tau^+(1)\bar{\tau}^-) \\ &= (x^+(0)\tau^+) (1)\bar{\tau}^- + \tau^+(1) (x^+(0)\bar{\tau}^-) \\ &= 0 + \tau^+(1)\bar{\tau}^+. \end{aligned}$$

Similarly, $\tau^-(1)\bar{\tau}^- = 2x^-$. □

3.2. Assumption (III). Lemma 3 implies

$$(16) \quad A(1)A = A(2)A = B(1)B = B(2)B = 0.$$

Thus super commutativity implies

Lemma 8. $[v(m), w(n)] = 0$ for all $v, w \in A$ for all $v, w \in B$.

Lemma 9. $u(1)v = 0$ for all $u \in \mathfrak{sl}_2$ and $v \in A \oplus B$.

Proof. The vector space $\mathfrak{sl}_2(1)A$ is spanned by the 6 vectors of the form $u(1)\tau^\pm$ for $u = x^\pm$ or h . Recall that $\tau^+(1)\bar{\tau}^+ = -2x^+$ from (15). Hence, by super associativity

$$\begin{aligned} x^+(1)\tau^+ &= -\frac{1}{2} \sum_{i \geq 0} (-1)^i \binom{1}{i} (\tau^+(1-i)\bar{\tau}^+(1+i) - \bar{\tau}^+(2-i)\tau^+(i)) \tau^+ \\ &= -\frac{1}{2} \tau^+(1)\bar{\tau}^+(1)\tau^+ + 0 = -\tau^+(1)x^+, \end{aligned}$$

using (16), $\bar{\tau}^+(2)\tau^+ \in \mathbb{C}1$ and that $\bar{\tau}^+(1)\tau^+ = -\tau^+(1)\bar{\tau}^+ = 2x^+$ by super skew symmetry. But, by super skew symmetry again we directly have $x^+(1)\tau^+ = +\tau^+(1)x^+$. Thus $x^+(1)\tau^+ = 0$. Furthermore, $(x^-(0))^k x^+(1)\tau^+ = 0$ for $k = 1, 2, 3$ results in the identities

$$h(1)\tau^+ - x^+(1)\tau^- = x^-(1)\tau^+ + h(1)\tau^- = x^-(1)\tau^- = 0.$$

We can then repeat a similar argument based on super associativity and skewsymmetry using $h = \tau^-(1)\bar{\tau}^+$ to show that $h(1)\tau^\pm = 0$. Thus $\mathfrak{sl}_2(1)A = 0$. By an identical argument we also find $\mathfrak{sl}_2(1)B = 0$. □

We next establish property (k):

Lemma 10. *We have*

$$\tau^\pm(2)\bar{\tau}^\pm = 0, \quad \tau^\pm(2)\bar{\tau}^\mp = \pm 2k\mathbf{1}.$$

Consequently, for $c = 6k$ we have

$$G^\pm(2)\bar{G}^\pm = 0, \text{ and } G^\pm(2)\bar{G}^\mp = \frac{c}{3}\mathbf{1}.$$

Proof. The image of $\Sigma \rightarrow A(2)B = \mathbb{C}\mathbf{1}$ is \mathfrak{sl}_2 -invariant, hence is 0. Then

$$\tau^\pm(2)\bar{\tau}^\pm = \tau^+(2)\bar{\tau}^- + \tau^-(2)\bar{\tau}^+ = 0.$$

The invariant $\tau^+(2)\bar{\tau}^- - \tau^-(2)\bar{\tau}^+$ is computed as follows: $h(1)\tau^\pm = h(1)\bar{\tau}^\pm = 0$ (Lemma 9) together with commutativity imply

$$(17) \quad [h(1), \tau^\pm(n)] = \pm \tau^\pm(n+1), \quad [h(1), \bar{\tau}^\pm(n)] = \pm \bar{\tau}^\pm(n+1).$$

Using (15) we thus find

$$2k\mathbf{1} = h(1)h = h(1)(\tau^\pm(1)\bar{\tau}^\mp) = \pm \tau^\pm(2)\bar{\tau}^\mp.$$

□

Lemma 11. *Let*

$$\sigma := \frac{1}{2}(\tau^+(0)\bar{\tau}^- - \tau^-(0)\bar{\tau}^+).$$

Then $\sigma(0)u = Tu$, $\sigma(1)u = u$ and $\sigma(2)u = 0$ for all $u \in \mathfrak{sl}_2$. In particular, σ is a nonzero \mathfrak{sl}_2 -invariant.

Proof. σ is an \mathfrak{sl}_2 -invariant in $A(0)B \subseteq V_2$ because it generates the image of Λ under the \mathfrak{sl}_2 -morphism defined by $A \otimes B \rightarrow A(0)B$.

$\sigma(2)\mathfrak{sl}_2 \in \mathbb{C}\mathbf{1}$ implies $\sigma(2)\mathfrak{sl}_2 = 0$ since $\mathbb{C}\mathbf{1}$ is a trivial \mathfrak{sl}_2 representation. By skewsymmetry, $\sigma(1)u = u(1)\sigma$ for all $u \in \mathfrak{sl}_2$. Hence, using (17) and Lemma 9, we find

$$\begin{aligned} \sigma(1)h &= \frac{1}{2}h(1)(\tau^+(0)\bar{\tau}^- - \tau^-(0)\bar{\tau}^+) \\ &= \frac{1}{2}(\tau^+(1)\bar{\tau}^- + \tau^-(1)\bar{\tau}^+) = h, \end{aligned}$$

from (15). This proves, in particular, that $\sigma \neq 0$. Since σ is \mathfrak{sl}_2 -invariant and h generates $A(1)B$ as an \mathfrak{sl}_2 -module, it follows that $\sigma(1)u = u$ for all $u \in \mathfrak{sl}_2$. Furthermore, from skew-symmetry we find that

$$\begin{aligned} \sigma(0)u &= -u(0)\sigma + T(u(1)\sigma) \\ &= 0 + T(\sigma(1)u) = Tu, \end{aligned}$$

using the \mathfrak{sl}_2 -invariance of σ .

□

Lemma 12. $\sigma(1)v = \frac{3}{2}v$ and $\sigma(2)v = 0$ for all $v \in A \oplus B$.

Proof. Using superassociativity we find

$$\sigma(1) = \frac{1}{2}[\tau^+(0), \bar{\tau}^-(1)] - \frac{1}{2}[\tau^-(0), \bar{\tau}^+(1)].$$

Hence

$$\begin{aligned}\sigma(1)\tau^+ &= \frac{1}{2}\tau^+(0)\bar{\tau}^-(1)\tau^+ - \frac{1}{2}\tau^-(0)\bar{\tau}^+(1)\tau^+ \\ &= -\frac{1}{2}\tau^+(0)h - \frac{1}{2}\tau^-(0)(2x^+) \\ &= \frac{1}{2}\tau^+ + \tau^+ = \frac{3}{2}\tau^+, \end{aligned}$$

using $\tau^\pm(0)\tau^\pm = 0$ (by assumption (III), (15) and super skew-symmetry). Using superassociativity, we similarly find

$$\sigma(2) = \frac{1}{2}[\tau^+(0), \bar{\tau}^-(2)] - \frac{1}{2}[\tau^-(0), \bar{\tau}^+(2)].$$

Hence

$$\sigma(2)\tau^+ = \frac{1}{2}\tau^+(0)\bar{\tau}^-(2)\tau^+ - \frac{1}{2}\tau^-(0)\bar{\tau}^+(2)\tau^+ = 0.$$

Similar results follow for τ^- by \mathfrak{sl}_2 -symmetry and for $\bar{\tau}^\pm$. \square

3.3. Assumption (IV). We next obtain properties (l) and (m).

Lemma 13. *We have*

$$\tau^\pm(0)\bar{\tau}^\pm = \mp Tx^\pm, \quad \tau^+(0)\bar{\tau}^- + \tau^-(0)\bar{\tau}^+ = Th.$$

Consequently,

$$G^\pm(0)\bar{G}^\pm = TJ^\pm, \text{ and } G^\pm(0)\bar{G}^\mp = \sigma \pm \frac{1}{2}TJ^0.$$

Proof. By assumption (IV), there exists $u \in \mathfrak{sl}_2$ such that $Tu \in A(0)B$. Since $A(0)B$ is an \mathfrak{sl}_2 -module and $T\mathfrak{sl}_2 \cong \mathfrak{sl}_2$ it follows that $T\mathfrak{sl}_2 \subseteq A(0)B$. Thus there is a morphism of \mathfrak{sl}_2 adjoint-modules $\Sigma \rightarrow T\mathfrak{sl}_2 \subseteq A(0)B$. In particular

$$\tau^-(0)\bar{\tau}^- = \kappa Tx^-,$$

for some $\kappa \neq 0$. Using (9) we have

$$L^U(1)\tau^-(0)\bar{\tau}^- = \tau^-(1)\bar{\tau}^- = 2x^-,$$

from (15). This implies

$$2x^- = L^U(1)\kappa L^U(-1)x^- = 2\kappa x^-.$$

Hence $\kappa=1$ and $\tau^-(0)\bar{\tau}^- = Tx^-$. The other relations follow by \mathfrak{sl}_2 -symmetry. \square

Lemma 14. *The modes of $\tau^\pm, \bar{\tau}^\pm$ satisfy the commutator relations*

$$(18) \quad [\tau^\pm(m), \bar{\tau}^\pm(n)] = \pm(n-m)x^\pm(m+n-1),$$

$$(19) \quad [\tau^\pm(m), \bar{\tau}^\mp(n)] = \pm\sigma(m+n) + \frac{1}{2}(m-n)h(m+n-1) \pm m(m-1)k\delta_{m+n+1,0}.$$

Proof. Properties (k)–(m), which we have already established, show that

$$\begin{aligned}\tau^\pm(0)\bar{\tau}^\pm &= \mp Tx^\pm, \quad \tau^\pm(1)\bar{\tau}^\pm = \mp 2x^\pm, \quad \tau^\pm(2)\bar{\tau}^\pm = 0, \\ \tau^\pm(0)\bar{\tau}^\mp &= \frac{1}{2}Th \pm \sigma, \quad \tau^\pm(1)\bar{\tau}^\mp = h, \quad \tau^\pm(2)\bar{\tau}^\mp = \pm 2k\mathbf{1},\end{aligned}$$

which imply the result from the super commutator formula. \square

Lemma 15. $\sigma(0)v = Tv$ for all $v \in A \oplus B$.

Proof. Using (19) we have

$$\sigma(0) = [\tau^+(0), \bar{\tau}^-(0)].$$

Thus we find using super skew-symmetry that

$$\begin{aligned} \sigma(0)\tau^+ &= \tau^+(0)\bar{\tau}^-(0)\tau^+ + 0 \\ &= \tau^+(0) (\tau^+(0)\bar{\tau}^- - T(\tau^+(1)\bar{\tau}^-)) \\ &= 0 - \tau^+(0)Th \\ &= (Th)(0)\tau^+ - T((Th)(1)\tau^+) + 0 \\ &= 0 + T\tau^+, \end{aligned}$$

using $(Th)(0) = 0$ and $(Th)(1) = -h(0)$. A similar argument applies to the remaining elements of $A \oplus B$. \square

Lemma 16. *For all $u \in A(1)B$ and $v \in A \oplus B$ we have*

$$(20) \quad [\sigma(m), u(n)] = -nu(m+n-1),$$

$$(21) \quad [\sigma(m), v(n)] = \left(\frac{1}{2}m - n\right)v(m+n-1).$$

Proof. Using Lemma 11 we have $\sigma(2)u=0$, $\sigma(1)u=u$ and $\sigma(0)u=Tu$ for $u \in \mathfrak{sl}_2$ the result (20) follows from the commutator formula. Likewise, (21) follows from $\sigma(2)v=0$, $\sigma(1)v=\frac{3}{2}v$ and $\sigma(0)v=Tv$ for all $v \in A \oplus B$ using Lemmas 12 and 15. \square

Lemma 17. *σ is a Virasoro vector for central charge $c=6k$.*

Proof. We have to check the relations

$$\sigma(0)\sigma = T\sigma, \quad \sigma(1)\sigma = 2\sigma, \quad \sigma(2)\sigma = 0, \quad \sigma(3)\sigma = 3k\mathbf{1}.$$

Using (21) we find

$$\begin{aligned} \sigma(1)\sigma &= \frac{1}{2} (\sigma(1)\tau^+(0)\bar{\tau}^- - \sigma(1)\tau^-(0)\bar{\tau}^+) \\ &= \frac{1}{2} \left(\frac{1}{2}\tau^+(0)\bar{\tau}^- + \tau^+(0)\frac{3}{2}\bar{\tau}^- - \frac{1}{2}\tau^-(0)\bar{\tau}^+ - \tau^-(0)\frac{3}{2}\bar{\tau}^+ \right) = 2\sigma, \\ \sigma(2)\sigma &= \frac{1}{2} (\sigma(2)\tau^+(0)\bar{\tau}^- - \sigma(2)\tau^-(0)\bar{\tau}^+) \\ &= \frac{1}{2} (\tau^+(1)\bar{\tau}^- + 0 - \tau^-(1)\bar{\tau}^+ - 0) = 0, \\ \sigma(3)\sigma &= \frac{1}{2} (\sigma(3)\tau^+(0)\bar{\tau}^- - \sigma(3)\tau^-(0)\bar{\tau}^+) \\ &= \frac{1}{2} \left(\frac{3}{2}\tau^+(2)\bar{\tau}^- + 0 - \frac{3}{2}\tau^-(2)\bar{\tau}^+ - 0 \right) = 3k\mathbf{1}. \end{aligned}$$

Lastly, by skew symmetry

$$\begin{aligned} \sigma(0)\sigma &= -\sigma(0)\sigma + T(\sigma(1)\sigma) \\ &= -\sigma(0)\sigma + 2T\sigma, \end{aligned}$$

so that $\sigma(0)\sigma = T\sigma$. \square

Thus Theorem 1 holds since all the defining relations for the $N=4, c=6k$ super conformal algebra $\mathcal{A}_{N=4}$ are satisfied.

Remark 18. If U is C_2 -cofinite and of strong CFT type then k is a positive integer by [DM] since $\mathfrak{sl}_2 \subseteq U_1$. We also note that the $N=4$ Virasoro vector σ (of Lemma 11) and ω^U (the Virasoro element of U) can be independent vectors of different central charges. Thus Theorem 1 is not a generating theorem for $N=4$ algebras (such as in [K] or [dS]) but rather describes the existence of a $N=4$ subalgebra of a given SVOA.

Finally we note that the automorphism group of $\mathcal{A}_{N=4}$ contains an involution g defined by

$$g : A \oplus B \rightarrow B \oplus A \\ (\tau^\pm, \bar{\tau}^\pm) \mapsto (\bar{\tau}^\pm, -\tau^\pm).$$

This follows by directly verifying that the defining relations (a)–(m) are preserved by g by use of super skew-symmetry i.e. $u(1)v = v(1)u$ and $u(0)v - v(0)u = -T(u(1)v) \in T\mathfrak{sl}_2$ for all $u \in A$ and $v \in B$. Furthermore, Assumption (IV) of Theorem 1 therefore has the following reformulation

Lemma 19. $T\mathfrak{sl}_2 \cap A(0)B \neq 0$ if and only if $A(0)B = B(0)A$. □

4. PROOF OF THEOREM 2

In this Subsection we assume the hypotheses and notation of Theorem 2, in particular V is contained in a super lattice VOA V_L (see Subsection 7.3 for the definition and relevant properties). In particular, $(V_L)_1$ is a reductive Lie algebra and each of its components is a simple Lie algebra of type ADE and of level 1.

Now by hypothesis (IV) of Theorem 2 our Lie algebra \mathfrak{sl}_2 contains a root of L . It follows that \mathfrak{sl}_2 is contained in one of the components of V_1 and therefore it also has level $k=1$. Adopting the notation of the previous Section, it follows that $(h, h)=2$ (cf. Lemma 4). Thus h is a root of L and we have $\mathfrak{sl}_2 = \langle h, e^{\pm h} \rangle$.

We will deduce Theorem 2 from Theorem 1. To this end, notice that states of weight $\frac{3}{2}$ in V_L are primary because L has no vectors of norm 1. Thus it suffices to take $U := V_L$ in Theorem 1 and show that hypotheses (I)–(IV) of Theorem 1 hold. Then Theorem 1 shows that V is the $N=4$ super conformal algebra of central charge $6k=6$. Parts (I) and (II) hold by assumption, so we only have to establish (III) and (IV).

We need some additional notation. Let $(\ , \) : L \times L \rightarrow \mathbb{Z}$ be the bilinear form on L . Note that this is *not* the notation used in the proof of Lemma 4, where $(\ , \)$ denoted the invariant bilinear form on \mathfrak{sl}_2 .

The vectors in L of norm n are denoted by

$$L_n := \{\alpha \in L \mid (\alpha, \alpha) = n\},$$

in particular L_2 is the root system of L . Fix a multiplicative bicharacter $\varepsilon : L \times L \rightarrow \{\pm 1\}$ that defines the central extension \widehat{L} occurring in the short exact sequence

$$1 \rightarrow \{\pm 1\} \rightarrow \widehat{L} \rightarrow L \rightarrow 0.$$

See Subsection 7.2 for more details on the ε -formalism, in particular for the justification that

$$(22) \quad \varepsilon(\alpha, \alpha) = \varepsilon(\alpha, -\alpha) = \begin{cases} -1 & \alpha \in L_2 \\ 1 & \alpha \in L_3 \cup L_4 \end{cases}$$

Recall (2) that $\tau^+, \bar{\tau}^+$ are highest weight vectors in A and B respectively. We have already mentioned that $U_{\frac{3}{2}}$ is spanned by states e^β ($\beta \in L_3$). Thus there are nonempty subsets $X, Y \subseteq L_3$ and scalars c_α, d_λ such that

$$(23) \quad \tau^+ := \sum_{\alpha \in X} c_\alpha e^\alpha, \quad \bar{\tau}^+ := \sum_{\lambda \in Y} d_\lambda e^\lambda.$$

Because $h(0)\tau^+ = \tau^+$ then we have $(h, \alpha) = 1$ ($\alpha \in X$), and similarly $(h, \lambda) = 1$ ($\lambda \in Y$).

As a result, we have the following useful facts that hold for $\alpha, \beta \in X$. $|(\alpha, \beta)| \leq 3$ by the Schwarz inequality, moreover

$$(\alpha, \beta) = \begin{cases} \pm 3 & \text{iff } \alpha = \pm \beta \\ 2 & \text{iff } \alpha - \beta = \gamma \text{ (root } \gamma \perp h) \\ -2 & \text{iff } \alpha + \beta = h \\ -1 & \text{iff } \alpha + \beta = h + \gamma \text{ (root } \gamma \perp h) \end{cases}$$

Identical formulas hold in case $\alpha, \beta \in Y$. We use these formulas in later calculations.

The elements $x^\pm \in \mathfrak{sl}_2$ may be identified (recall that $\varepsilon(h, h) = -1$ by (22)) as $x^\pm = \mp e^{\pm h}$. Because $x^-(0)\tau^+ = \tau^-$ we have

$$\tau^- = \sum_{\alpha \in X} c_\alpha e^{-h}(0)e^\alpha = \sum_{\alpha \in X} c_\alpha \varepsilon(h, \alpha) e^{\alpha-h},$$

and similarly

$$\bar{\tau}^- = \sum_{\lambda \in Y} d_\lambda \varepsilon(h, \lambda) e^{\lambda-h}.$$

We now consider the consequences of assumption (III) of Theorem 2.

Lemma 20. $A(1)A = 0$ implies

$$\begin{aligned} (a) \quad & \sum_{\alpha \in X} c_\alpha c_{h+\gamma-\alpha} \varepsilon(\alpha, h+\gamma) = 0 \quad (\text{each root } \gamma \perp h), \\ (b) \quad & \sum_{\alpha \in X} c_\alpha c_{h-\alpha} \varepsilon(h, \alpha) \alpha = 0, \end{aligned}$$

with a corresponding statement concerning $B(1)B = 0$.

Proof. $A(1)A$ contains the element $\tau^+(1)\tau^-$, which is equal to

$$\begin{aligned}
 \sum_{\alpha, \beta \in X} c_\alpha c_\beta \varepsilon(h, \beta) e^\alpha(1) e^{\beta-h} &= \sum_{(\alpha, \beta-h)=-2, -3} c_\alpha c_\beta \varepsilon(h, \beta) e^\alpha(1) e^{\beta-h} \\
 &= \sum_{\alpha+\beta=h} c_\alpha c_\beta \varepsilon(h, \beta) \alpha + \sum_{(\alpha, \beta)=-1} c_\alpha c_\beta \varepsilon(h, \beta) \varepsilon(\alpha, \beta-h) e^{\alpha+\beta-h} \\
 &= \sum_{\alpha+\beta=h} c_\alpha c_\beta \varepsilon(h, \beta) \alpha + \sum_{\gamma \perp h} \left\{ \sum_{\alpha+\beta=h+\gamma} c_\alpha c_\beta \varepsilon(h, \beta) \varepsilon(\alpha, \gamma-\alpha) \right\} e^\gamma \\
 &= \sum_{\alpha+\beta=h} c_\alpha c_\beta \varepsilon(h, \beta) \alpha - \sum_{\gamma \perp h} \sum_{\alpha+\beta=h+\gamma} c_\alpha c_\beta \varepsilon(h, \gamma+\alpha) \varepsilon(\alpha, \gamma) e^\gamma \\
 &= \sum_{\alpha+\beta=h} c_\alpha c_\beta \varepsilon(h, \beta) \alpha - \sum_{\gamma \perp h} \varepsilon(h, \gamma) \sum_{\alpha+\beta=h+\gamma} c_\alpha c_\beta \varepsilon(h, \alpha) \varepsilon(\alpha, \gamma) e^\gamma \\
 &= - \sum_{\alpha} c_\alpha c_{h-\alpha} \varepsilon(h, \alpha) \alpha + \sum_{\gamma \perp h} \varepsilon(h, \gamma) \sum_{\alpha} c_\alpha c_{h+\gamma-\alpha} \varepsilon(\alpha, h+\gamma) e^\gamma,
 \end{aligned}$$

using $\varepsilon(h, \beta) = \varepsilon(h, h-\alpha) = -\varepsilon(h, \alpha)$. Hence $A(1)A=0$ if and only if

$$\sum_{\alpha \in X} c_\alpha c_{h+\gamma-\alpha} \varepsilon(\alpha, h+\gamma) = 0,$$

for each root $\gamma \perp h$, and

$$\sum_{\alpha} c_\alpha c_{h-\alpha} \varepsilon(h, \alpha) \alpha = 0.$$

A similar analysis applies for $B(1)B = 0$. □

Lemma 21. *We have $A(0)A=B(0)B=0$.*

Proof. We prove that $A(0)A=0$. The proof that $B(0)B=0$ is similar. Assume, then, that $A(0)A \neq 0$. Because $A(1)A=0$, by assumption (III), then $A(0)A$ has dimension ≤ 3 by super skew-symmetry, indeed because we are assuming that $A(0)A \neq 0$ then the image of $A \otimes A \rightarrow A(0)A$ is the adjoint module for \mathfrak{sl}_2 . Now it follows that $0 \neq \tau^+(0)\tau^+ \in A(0)A$. But we also have

$$\begin{aligned}
 \tau^+(0)\tau^+ &= \sum_{\alpha, \beta \in X} c_\alpha c_\beta e^\alpha(0) e^\beta = \sum_{(\alpha, \beta)=-1} c_\alpha c_\beta \varepsilon(\alpha, \beta) e^{\alpha+\beta} + \sum_{(\alpha, \beta)=-2} c_\alpha c_\beta \varepsilon(\alpha, \beta) \alpha (-1) e^{\alpha+\beta} \\
 &= \sum_{\gamma} \left\{ \sum_{\alpha+\beta=h+\gamma} c_\alpha c_\beta \varepsilon(\alpha, h+\gamma) \right\} e^{h+\gamma} + \left\{ \sum_{\alpha+\beta=h} c_\alpha c_\beta \varepsilon(\alpha, h) \alpha \right\} (-1) e^h = 0,
 \end{aligned}$$

where we used Lemma 20. This contradiction completes the proof of the Lemma. □

This establishes assumption (III) of Theorem 1.

We now consider consequences of assumptions (II) and (IV) of Theorem 2.

Lemma 22. *$A(1)B \cong \mathfrak{sl}_2$ with $\mathfrak{sl}_2 = \langle h, e^{\pm h} \rangle$ if and only if*

$$(a) \quad h = - \sum_{\alpha \in X} c_\alpha d_{h-\alpha} \varepsilon(h, \alpha) \alpha,$$

$$(b) \quad \sum_{\gamma \perp h} \varepsilon(h, \gamma) \left\{ \sum_{\alpha \in X} c_\alpha d_{h+\gamma-\alpha} \varepsilon(\alpha, h+\gamma) \right\} e^\gamma = 0,$$

where the γ sum is taken over each root $\gamma \perp h$.

Proof. By calculations similar to those of Lemma 20 we have

$$\begin{aligned}\tau^+(1)\bar{\tau}^- &= \sum_{\alpha \in X, \lambda \in Y} c_\alpha d_\lambda \varepsilon(h, \lambda) e^\alpha(1) e^{\lambda-h} = \sum_{(\alpha, \lambda-h)=-2, -3} c_\alpha d_\lambda \varepsilon(h, \lambda) e^\alpha(1) e^{\lambda-h} \\ &= - \sum_{\alpha \in X} c_\alpha d_{h-\alpha} \varepsilon(h, \alpha) \alpha + \sum_{\gamma \perp h} \varepsilon(h, \gamma) \sum_{\alpha \in X} c_\alpha d_{h+\gamma-\alpha} \varepsilon(\alpha, h+\gamma) e^\gamma.\end{aligned}$$

But from (15) we have that $h = \tau^+(1)\bar{\tau}^-$ iff (a) and (b) hold. Note that taking the inner product of (a) with h implies

$$(24) \quad \sum_{\alpha \in X} c_\alpha d_{h-\alpha} \varepsilon(h, \alpha) = -2.$$

This implies

$$\begin{aligned}\tau^-(1)\bar{\tau}^+ &= \sum_{\alpha \in X, \lambda \in Y} c_\alpha d_\lambda \varepsilon(h, \alpha) e^{\alpha-h}(1) e^\lambda = \sum_{(\alpha-h, \lambda)=-2, -3} c_\alpha d_\lambda \varepsilon(h, \alpha) e^{\alpha-h}(1) e^\lambda \\ &= \sum_{\alpha \in X} c_\alpha d_{h-\alpha} \varepsilon(h, \alpha) (\alpha - h) + \sum_{\gamma \perp h} \varepsilon(h, \gamma) \sum_{\alpha \in X} c_\alpha d_{h+\gamma-\alpha} \varepsilon(\alpha, h+\gamma) e^\gamma \\ &= -h + 2h + 0 = h,\end{aligned}$$

iff (a) and (b) hold. The remaining relations in (15) follow from \mathfrak{sl}_2 symmetry. \square

Lemma 23. $A(1)B \cong \mathfrak{sl}_2$ implies $T \mathfrak{sl}_2 \cap A(0)B \neq 0$.

Proof. $A(0)B$ contains the element

$$\begin{aligned}\tau^+(0)\bar{\tau}^+ &= \sum_{(\alpha, \lambda)=-1, -2} c_\alpha d_\lambda e^\alpha(0) e^\lambda \\ &= -\frac{1}{2} \left\{ \sum_{\alpha \in X} c_\alpha d_{h-\alpha} \varepsilon(h, \alpha) \right\} h(-1) e^h = h(-1) e^h = T e^{-h},\end{aligned}$$

by (24) of Lemma 22. Thus $T \mathfrak{sl}_2 \cap A(0)B \neq 0$. \square

This completes the proof of hypothesis(IV) of Theorem 1, and with it the proof of Theorem 2.

5. $N=2$ SUPERCONFORMAL ALGEBRAS

The $N=2$ SVOA $\mathcal{A}_{N=2}$ of central charge c is generated by a pair of states τ^\pm of conformal weight $\frac{3}{2}$ satisfying the non-zero relations e.g. [K]

$$\begin{aligned}(i) \quad \tau^\pm(2)\tau^\mp &= \frac{c}{3}\mathbf{1}, & (ii) \quad \tau^\pm(1)\tau^\mp &= \pm h, & (iii) \quad \tau^\pm(0)\tau^\mp &= \omega \pm \frac{1}{2}T, \\ (iv) \quad h(0)\tau^\pm &= \pm \tau^\pm, & (v) \quad h(1)h &= \frac{c}{3}\mathbf{1},\end{aligned}$$

together with the standard Virasoro relations between τ^\pm, h and the Virasoro vector ω of central charge c . Similarly to Theorem 1 we have

Theorem 24. *Let U be a SVOA of CFT-type. Let $V \subseteq U$ be the subalgebra generated by 2 primary vectors τ^\pm of weight $\frac{3}{2}$ in U , so that*

$$V = \mathbb{C}\mathbf{1} \oplus V_{\frac{1}{2}} \oplus V_1 \oplus V_{\frac{3}{2}} \oplus \dots$$

is a conformally graded subspace of U . Assume that:

- (I) $h(0)\tau^\pm = \pm \tau^\pm$ where $h := \tau^+(1)\tau^-$,
- (II) $\tau^\pm(0)\tau^\pm = 0$.

Then $V \cong \mathcal{A}_{N=2}$ with central charge $c=6k$ where $h(1)h = 2k\mathbf{1}$.

Proof. We sketch the proof, which is similar in many respects to that for Theorem 1. We firstly note that $u(1)v = -v(1)u$ for all $u, v \in \langle \tau^\pm \rangle$ by super skew-symmetry. Thus Assumptions (I) and (II) above imply the properties (ii), (iv) and (v) for $c=6k$. As for Lemma 9, we find $h(1)\tau^\pm = 0$ which in turn implies (as in Lemma 10) that $\tau^\pm(2)\tau^\mp = 2k\mathbf{1}$. Thus property (i) holds.

Using super skew-symmetry we have

$$\tau^+(0)\tau^- = \tau^-(0)\tau^+ - T(\tau^-(1)\tau^+) = \tau^-(0)\tau^+ + Th.$$

Thus, in this case, we define

$$(25) \quad \sigma := \frac{1}{2} (\tau^+(0)\tau^- + \tau^-(0)\tau^+),$$

so that $\tau^\pm(0)\tau^\mp = \sigma \pm \frac{1}{2}Th$. Hence

$$(26) \quad [\tau^+(m), \tau^-(n)] = \sigma(m+n) + \frac{1}{2}(m-n)h(m+n-1) + m(m-1)k\delta_{m+n+1,0}.$$

It remains to show that σ is a Virasoro vector of central charge $c=6k$. As in Lemma 11, we find $\sigma(0)h = Th$, $\sigma(1)h = h$ and $\sigma(2)h = 0$. (26) implies

$$\sigma(0) = [\tau^+(0), \tau^-(0)], \quad \sigma(1) = [\tau^+(0), \tau^-(1)] + \frac{1}{2}h(0), \quad \sigma(2) = [\tau^+(0), \tau^-(2)] + h(1),$$

from which it follows that $\sigma(0)\tau^+ = T\tau^+$, $\sigma(1)\tau^+ = \frac{3}{2}\tau^+$ and $\sigma(2)\tau^+ = 0$ (cf. Lemmas 12 and 15). Similar results follow for $\sigma(n)\tau^-$ for $n = 0, 1, 2$. Hence (cf. Lemma 16)

$$[\sigma(m), \tau^\pm(n)] = \left(\frac{1}{2}m-n\right) \tau^\pm(m+n-1),$$

which implies that σ is a Virasoro vector of central charge $6k$ (cf. Lemma 17). \square

6. EXAMPLES

We provide some constructions of $N=4$ and $N=2$ subalgebras of a lattice SVOA V_L for an odd lattice L . These examples illustrate Theorems 1, 2 and 24. Throughout, we let L_n denote the set of lattice vectors in L of norm n .

6.1. Example 1. Consider the lattice SVOA V_L for $L = \mathbb{Z}^6$ – the well-known rank 12 free fermion construction. Let $L = \mathbb{Z}^6$ be generated by $\gamma_1, \dots, \gamma_6 \in L_1$ with $(\gamma_i, \gamma_j) = \delta_{ij}$. Then V_L is generated by 12 weight $\frac{1}{2}$ fermion vectors $e^{\pm\gamma_i}$.

We firstly note from Section 7.2 that

$$\varepsilon(\gamma_i, \gamma_j) = \begin{cases} -\varepsilon(\gamma_j, \gamma_i) & \text{for } i \neq j, \\ -1 & \text{for } i = j. \end{cases}$$

In addition, for convenience, we choose $\varepsilon(\gamma_1, \gamma_2) = 1$ so that $\varepsilon(\gamma_2, \gamma_1) = -1$.

Define \mathfrak{sl}_2 generators $h := \gamma_1 + \gamma_2$, $x^\pm := \mp e^{\pm h} \in (V_{\mathbb{Z}^6})_1$ where $h(1)h = 2\mathbf{1}$ i.e. $k=1$ in (13). Then $\langle e^{\gamma_1}, e^{-\gamma_2} \rangle$ and $\langle e^{\gamma_2}, e^{-\gamma_1} \rangle$ form a pair of \mathfrak{sl}_2 -representations, where using Subsection 7.3, we find

$$(27) \quad \begin{aligned} h(0)e^{\pm\gamma_1} &= \pm e^{\pm\gamma_1}, & h(0)e^{\pm\gamma_2} &= \pm e^{\pm\gamma_2}, \\ x^\pm(0)e^{\mp\gamma_1} &= \mp e^{\pm\gamma_2}, & x^\pm(0)e^{\mp\gamma_2} &= \pm e^{\pm\gamma_1}. \end{aligned}$$

Define $a, \bar{a}, b, \bar{b} \in (V_{\mathbb{Z}^6})_1$ by

$$\begin{aligned} a &= \frac{1}{\sqrt{2}}(\gamma_3 + i\gamma_4), & \bar{a} &= \frac{1}{\sqrt{2}}(\gamma_3 - i\gamma_4), \\ b &= \frac{1}{\sqrt{2}}(\gamma_5 + i\gamma_6), & \bar{b} &= \frac{1}{\sqrt{2}}(\gamma_5 - i\gamma_6), \end{aligned}$$

which satisfy non-zero relations $a(1)\bar{a} = b(1)\bar{b} = \mathbf{1}$. Lastly, define $\tau^\pm, \bar{\tau}^\pm$ by

$$\begin{aligned} \tau^+ &= a(-1)e^{\gamma_1} + b(-1)e^{\gamma_2}, & \tau^- &= a(-1)e^{-\gamma_2} - b(-1)e^{-\gamma_1}, \\ \bar{\tau}^+ &= \bar{a}(-1)e^{\gamma_2} - \bar{b}(-1)e^{\gamma_1}, & \bar{\tau}^- &= -\bar{a}(-1)e^{-\gamma_1} - \bar{b}(-1)e^{-\gamma_2}. \end{aligned}$$

We now show that the sub-SVOA generated by $\tau^\pm, \bar{\tau}^\pm$ is isomorphic to $\mathcal{A}_{N=4}$ for central charge $c=6$ by use of Theorem 1. $\tau^\pm, \bar{\tau}^\pm$ are clearly primary vectors of weight $\frac{3}{2}$. It is straightforward to confirm (4) and (5) by using (27). Thus Axiom (I) of Theorem 1 holds.

In order to confirm Axioms (II)–(IV) of Theorem 1 we note, using superassociativity, that for all $u, v \in \{a, b\}$ and $\lambda, \nu \in \{\pm\gamma_1, \pm\gamma_2\}$

$$(28) \quad (u(-1)e^\lambda)(1)v(-1)e^\nu = \langle\langle u, v \rangle\rangle e^\lambda(-1)e^\nu,$$

$$(29) \quad (u(-1)e^\lambda)(0)v(-1)e^\nu = u(-1)v(-1)e^\lambda(0)e^\nu + \langle\langle u, v \rangle\rangle e^\lambda(-2)e^\nu,$$

where $u(1)v = \langle\langle u, v \rangle\rangle \mathbf{1}$. (28) and Subsection 7.3 imply that

$$\begin{aligned} \tau^+(1)\bar{\tau}^- &= -\varepsilon(\gamma_1, -\gamma_1)\gamma_1(-1)\mathbf{1} - \varepsilon(\gamma_2, -\gamma_2)\gamma_2(-1)\mathbf{1} = h, \\ \tau^-(1)\bar{\tau}^+ &= \varepsilon(-\gamma_2, \gamma_2)(-\gamma_2)(-1)\mathbf{1} + \varepsilon(-\gamma_1, \gamma_1)(-\gamma_1)(-1)\mathbf{1} = h. \end{aligned}$$

Therefore $\tau^+(1)\bar{\tau}^+ = -2x^+$ and $\tau^-(1)\bar{\tau}^- = 2x^-$ using $x^\pm(0)h = \mp 2x^\pm$ and hence $A(1)B \cong \mathfrak{sl}_2$ i.e. Axiom (II) holds. Axiom (III) follows from

$$\tau^+(0)\tau^- = -\varepsilon(-\gamma_1, -\gamma_1)a(-1)b(-1)\mathbf{1} + \varepsilon(\gamma_2, -\gamma_2)b(-1)a(-1)\mathbf{1} = 0.$$

using (29). By skew-symmetry $\tau^-(0)\tau^+ = \tau^+(0)\tau^- = 0$ and so $A(0)A = 0$ using \mathfrak{sl}_2 symmetry. A similar argument applies showing that $B(0)B = 0$. Lastly

$$\tau^+(0)\bar{\tau}^+ = -(\gamma_1 + \gamma_2)(-1)e^{\gamma_1 + \gamma_2} = -Tx^+,$$

so that Axiom (IV) holds. Hence the SVOA generated by $\tau^\pm, \bar{\tau}^\pm$ is isomorphic to $\mathcal{A}_{N=4}$ for central charge $c=6k = 6$ by Theorem 1.

To finish, we show that $\sigma=\omega$, the standard V_L Virasoro vector of central charge 6. From (29) we find

$$\begin{aligned}\sigma &= \frac{1}{2} (\tau^+(0)\bar{\tau}^- - \tau^-(0)\bar{\tau}^+) \\ &= -\frac{1}{2} \left(\varepsilon(\gamma_1, -\gamma_1)a(-1)\bar{a} + e^{\gamma_1}(-2)e^{-\gamma_1} + \varepsilon(\gamma_2, -\gamma_2)b(-1)\bar{b} + e^{\gamma_2}(-2)e^{-\gamma_2} \right. \\ &\quad \left. + \varepsilon(-\gamma_2, \gamma_2)a(-1)\bar{a} + e^{-\gamma_2}(-2)e^{\gamma_2} + \varepsilon(-\gamma_1, \gamma_1)b(-1)\bar{b} + e^{-\gamma_1}(-2)e^{\gamma_1} \right) \\ &= a(-1)\bar{a} + b(-1)\bar{b} + \frac{1}{2} (\gamma_1(-1)\gamma_1 + \gamma_2(-1)\gamma_2) = \frac{1}{2} \sum_{i=1}^6 \gamma_i(-1)\gamma_i = \omega.\end{aligned}$$

Thus we conclude that

Proposition 25. $V_{\mathbb{Z}^6}$ contains an $N=4$ superconformal subalgebra with the standard lattice Virasoro vector for central charge $c=6$.

6.2. Example 2. Let $\alpha_1, \dots, \alpha_6$ be an orthogonal basis for \mathbb{R}^6 consisting of vectors of norm 3, and let L be the lattice spanned by the α_i together with

$$h := \frac{1}{3}(\alpha_1 + \dots + \alpha_6) \in L_2.$$

Then L is an odd, positive-definite, integral lattice with *theta function*

$$\theta_L(\tau) = 1 + 2q + 24q^{\frac{3}{2}} + \dots$$

In particular, there are no vectors of norm 1, and $\pm h$ are the only roots.

Proposition 26. V_L contains an $N=4$ superconformal subalgebra A such that the Virasoro vector of A is the standard lattice Virasoro vector of central charge $c=6$.

Proof. Let $X \subseteq L_3$ consist of the 6 vectors α_i . In the formalism of Section 4, especially (23), we take Y to consist of the vectors $\{h - \alpha_i\}$, so that the four generating states of A of weight $\frac{3}{2}$ will be chosen to take the form

$$\begin{aligned}\tau^+ &:= \sum_{\alpha \in X} c_\alpha e^\alpha, & \tau^- &:= \sum_{\alpha \in X} c_\alpha \varepsilon(h, \alpha) e^{\alpha-h}, \\ \bar{\tau}^+ &:= \sum_{\alpha \in X} d_{h-\alpha} e^{h-\alpha}, & \bar{\tau}^- &:= - \sum_{\alpha \in X} d_{h-\alpha} \varepsilon(h, \alpha) e^{-\alpha}.\end{aligned}$$

We show that the hypotheses of Theorem 2 hold for certain choices of scalars $c_\alpha, d_{h-\alpha}$, ($\alpha \in X$). Conditions (a) and (b) of Lemma 20 and condition (b) of Lemma 22 automatically hold since $X \cap Y = 0$ and $\pm h$ are the only roots in L . We may check by direct calculation that these facts *imply* the assumptions of these two Lemmas (cf. the proofs of the Lemmas), i.e., $A(1)A = B(1)B = 0$. Now choose the scalars $c_\alpha, d_{h-\alpha}$ so that

$$c_\alpha d_{h-\alpha} \varepsilon(h, \alpha) = -\frac{1}{3}$$

for each $\alpha_i \in X$, implying condition (a) of Lemma 22. Hence $A(1)B \cong \mathfrak{sl}_2$.

Because $\pm h$ are the only roots of L , there is a unique simple component of the Lie algebra $(V_L)_1$, and it is isomorphic to sl_2 . It follows that this component is

our Lie algebra $A(1)B = \mathfrak{sl}_2$, and in particular $h \in \mathfrak{sl}_2$ and $A(1)B = \langle h, e^{\pm h} \rangle$. It is then straightforward to check that A and B are indeed vector representations of \mathfrak{sl}_2 , so that all hypotheses of Theorem 2 are satisfied. This completes the proof that $\tau^{\pm}, \bar{\tau}^{\pm}$ generate an $N=4$ subalgebra of V_L with $c=6$.

The Virasoro vector in this example is (cf. Lemmas 11, 17)

$$\begin{aligned}
\sigma &= \frac{1}{2}(\tau^+(0)\bar{\tau}^- - \tau^-(0)\bar{\tau}^+) \\
&= -\frac{1}{2} \sum_{\alpha, \beta \in X} \left\{ c_{\alpha} d_{h-\beta} \varepsilon(h, \beta) e^{\alpha}(0) e^{-\beta} + c_{\alpha} d_{h-\beta} \varepsilon(h, \alpha) e^{\alpha-h}(0) e^{h-\beta} \right\} \\
&= -\frac{1}{2} \sum_{\alpha \in X} \left\{ c_{\alpha} d_{h-\alpha} \varepsilon(h, \alpha) e^{\alpha}(0) e^{-\alpha} + c_{\alpha} d_{h-\alpha} \varepsilon(h, \alpha) e^{\alpha-h}(0) e^{h-\alpha} \right\} \\
&= \frac{1}{12} \sum_{\alpha \in X} \left(\alpha(-2) + \alpha(-1)^2 + (\alpha - h)(-2) + (\alpha - h)(-1)^2 \right) \mathbf{1} \\
&= \frac{1}{12} \sum_{\alpha \in X} \left(2\alpha(-1)^2 + h(-1)^2 - 2\alpha(-1)h(-1) + 2\alpha(-2) - h(-2) \right) \mathbf{1} \\
&= \frac{1}{6} \sum_{\alpha} \alpha(-1)\alpha.
\end{aligned}$$

This is indeed the standard Virasoro element of V_L , and the proof is complete. \square

We next consider three examples of $N=2$ superconformal subalgebras of odd lattice SVOAS, illustrating Theorem 24.

6.3. Example 3. Consider the lattice SVOA $V_{\mathbb{Z}^3}$, the rank 6 free fermion construction. Let $L = \mathbb{Z}^3$ be generated by $\gamma_1, \gamma_2, \gamma_3 \in L_1$ with $(\gamma_i, \gamma_j) = \delta_{ij}$. Define $h, a^{\pm} \in (V_{\mathbb{Z}^3})_1$ by

$$h = \gamma_1, \quad a^{\pm} = \frac{1}{\sqrt{2}}(\gamma_2 \pm i\gamma_3),$$

with $a^+(1)a^-=1$. Lastly, define $\tau^{\pm} = a^{\pm}(-1)e^{\pm\gamma_1}$.

Using (29) and (28) we find that Axioms (I) and (II) of Theorem 24 hold. Since $h(1)h=1$, the central charge is $c=3$. Using (25) one finds that

$$\sigma = a^+(-1)a^- + \frac{1}{2}\gamma_1(-1)\gamma_1 = \frac{1}{2} \sum_{i=1}^3 \gamma_i(-1)\gamma_i,$$

the standard lattice Virasoro vector for $V_{\mathbb{Z}^3}$. This establishes

Proposition 27. $V_{\mathbb{Z}^3}$ contains an $N=2$ superconformal subalgebra with the standard lattice Virasoro vector for central charge $c=3$.

6.4. Example 4. In this example we show that every odd lattice SVOA for which $L_3 \neq \emptyset$ contains an $N=2$ SVOA with central charge $c=1$.

Proposition 28. Let $\gamma \in L_3$ and define $\tau^{\pm} := \frac{1}{\sqrt{3}}e^{\pm\gamma}$ and $h := \frac{1}{3}\gamma$. Then τ^{\pm} generate an $N=2$ subalgebra of V_L with Virasoro vector $\omega = \frac{1}{6}\gamma(-1)\gamma$ and $c=1$.

Proof. We find $h=\tau^\pm(1)\tau^\mp$ with $h(0)\tau^\pm=\pm\tau^\pm$ and $\tau^\pm(0)\tau^\pm=0$. Thus Axioms (I) and (II) of Theorem 24 hold, so τ^\pm generate an $N=2$ superconformal algebra. Since $h(1)h=\frac{1}{3}\mathbf{1}$, the central charge is $c=1$ with Virasoro vector $\frac{1}{6}\gamma(-1)\gamma$ from (25). \square

6.5. Example 5.

Proposition 29. *Let $\alpha, \beta \in L_3$ with $(\alpha, \beta)=1$, and let λ, μ be nonzero scalars. Define*

$$\begin{aligned} \tau^+ &= \frac{1}{2}(\lambda e^\alpha + \mu e^\beta), \quad \tau^- = \frac{1}{2}(\lambda^{-1}e^{-\alpha} + \mu^{-1}e^{-\beta}), \quad h = \frac{1}{4}(\alpha + \beta), \\ \omega &= \frac{1}{8} \left\{ \alpha(-1)\alpha + \beta(-1)\beta + \frac{2\lambda}{\mu}\varepsilon(\alpha, \beta)e^{\alpha-\beta} + \frac{2\mu}{\lambda}\varepsilon(\beta, \alpha)e^{\beta-\alpha} \right\}. \end{aligned}$$

Then τ^\pm generate an $N=2$ subalgebra of V_L with $c=\frac{3}{2}$.

Proof. Axioms (I) and (II) of Theorem 24 are easily seen to hold. $h(1)h = \frac{1}{2}\mathbf{1}$ implies the central charge $c=\frac{3}{2}$ and ω is as given on applying (25). \square

7. APPENDICES

7.1. Axioms for super VOAs. The underlying Fock space is a $\frac{1}{2}\mathbb{Z}$ -graded \mathbb{C} -linear super vector space

$$V = \bigoplus_{k \in \frac{1}{2}\mathbb{Z}} V_k,$$

with *parity* operator $p(u) = 2k \bmod 2$ for $u \in V_k$. Each state $u \in V$ has a vertex operator $Y(u, z) = \sum_{n \in \mathbb{Z}} u(n)z^{-n-1}$; $u(n) \in \text{End}(V)$ is the n^{th} mode of u .

There is a distinguished *vacuum state* $\mathbf{1} \in V_0$ with vertex operator $Y(\mathbf{1}, z) = \text{Id}_V$; and a distinguished *Virasoro state* $\omega \in V_2$ with vertex operator

$$Y(\omega, z) = \sum_{n \in \mathbb{Z}} L(n)z^{-n-2},$$

whose modes satisfy the Virasoro relations with *central charge* c :

$$[L(m), L(n)] = (m-n)L(m+n) + \frac{1}{2} \binom{m+1}{3} \delta_{m+n,0} c \text{Id}_V.$$

We distinguish the endomorphism $T \in \text{End}(V)$ defined by $T(u) := u(-2)\mathbf{1} = L(-1)u$. The $\frac{1}{2}\mathbb{Z}$ grading is determined by $L(0)$ with $L(0)u = ku$ for $u \in V_k$.

Modes satisfy the following axioms, the third being the super Jacobi identity:

- (a) $u(n)v = 0$ for all $n \geq n_0$,
- (b) $u(-1)\mathbf{1} = u$; $u(n)\mathbf{1} = 0$ for $n \geq 0$,
- (c) $\forall r, s, t \in \mathbb{Z}$,

$$\begin{aligned} \sum_{i \geq 0} \binom{r}{i} (u(t+i)v)(r+s-i)w = \\ \sum_{i \geq 0} (-1)^i \binom{t}{i} \{ u(r+t-i)v(s+i) - (-1)^{t+p(u)p(v)} v(s+t-i)u(r+i) \} w, \end{aligned}$$

for all $u, v, w \in V$. The special cases $t=0, r=0$ give respectively super commutativity

$$u(r)v(s) - (-1)^{p(u)p(v)}v(s)u(r) = \sum_{i \geq 0} \binom{r}{i} (u(i)v)(r+s-i),$$

and super associativity

$$(u(t)v)(s) = \sum_{i \geq 0} (-1)^i \binom{t}{i} \{u(t-i)v(s+i) - (-1)^{t+p(u)p(v)}v(s+t-i)u(i)\}.$$

Taking $r=-1, s=0, w=1$ leads to super skew-symmetry

$$v(t)u = -(-1)^{t+p(u)p(v)} \sum_{i \geq 0} (-1)^i T^i u(t+i)v.$$

7.2. The ε -formalism. Fix a finitely generated free abelian group L . We are interested in groups \widehat{L} which are *central extensions of L by \mathbb{Z}_2* . So there is a short exact sequence of groups

$$(30) \quad 1 \rightarrow \{\pm 1\} \rightarrow \widehat{L} \xrightarrow{\pi} L \rightarrow 0,$$

and \widehat{L} can be identified with $L \times \{\pm 1\}$ as a set, with multiplication

$$(\alpha, e)(\beta, f) = (\alpha + \beta, \varepsilon(\alpha, \beta)ef) \quad (\alpha, \beta \in L, e, f \in \{\pm 1\}),$$

where

$$\varepsilon: L \times L \rightarrow \{\pm 1\}.$$

We may, and shall, take ε to be *bimultiplicative*, i.e.,

$$\varepsilon(\alpha + \beta, \gamma) = \varepsilon(\alpha, \gamma)\varepsilon(\beta, \gamma), \quad \varepsilon(\alpha, \beta + \gamma) = \varepsilon(\alpha, \beta)\varepsilon(\alpha, \gamma).$$

This ensures that $\varepsilon \in Z^2(L, \{\pm 1\})$ is a *2-cocycle* and that multiplication in L is *associative*. In particular we note that $\varepsilon(\alpha, 0) = \varepsilon(0, \alpha) = 1$ and $\varepsilon(\alpha, \beta) = \varepsilon(\alpha, -\beta) = \varepsilon(-\alpha, \beta)$.

If L is a positive-definite integral lattice with bilinear form $(\ , \)$, then we may further choose ε ([K] P. 155) so that it satisfies

$$\varepsilon(\alpha, \beta)\varepsilon(\beta, \alpha) = (-1)^{(\alpha, \beta) + (\alpha, \alpha)(\beta, \beta)}, \quad \varepsilon(\alpha, \alpha) = (-1)^{((\alpha, \alpha) + (\alpha, \alpha)^2)/2}.$$

Remark 30. Depending on context, various choices for ε are used in the literature, although they give equivalent theories. The one used in [FLM], for example, is different to the one we generally use here.

7.3. Super lattice theories. Let L be a positive-definite integral lattice equipped with a bilinear form $(\ , \)$, with ε as in Subsection 7.2. The *twisted group algebra* $\mathbb{C}^\varepsilon[L]$ has basis $e^\alpha (\alpha \in L)$ and multiplication

$$e^\alpha e^\beta = \varepsilon(\alpha, \beta) e^{\alpha + \beta} \quad (\alpha, \beta \in L).$$

It is $\frac{1}{2}\mathbb{Z}$ -graded by $wt(e^\alpha) := \frac{1}{2}(\alpha, \alpha)$.

There are Lie algebras (the first is abelian)

$$\mathfrak{h} := \mathbb{C} \otimes_{\mathbb{Z}} L, \quad \widehat{\mathfrak{h}} := \mathfrak{h} \otimes \mathbb{C}[t, t^{-1}] \oplus \mathbb{C}c, \quad \mathfrak{h}^+ = \mathfrak{h} \otimes t\mathbb{C}[t], \quad \widehat{\mathfrak{h}}^- = \mathfrak{h} \otimes t^{-1}\mathbb{C}[t^{-1}],$$

with brackets $[x \otimes t^m, y \otimes t^n] = (x, y)m\delta_{m+n, 0}c$, $[c, \widehat{\mathfrak{h}}] = 0$, and an induced $\widehat{\mathfrak{h}}$ -module

$$M(1) = U(\widehat{\mathfrak{h}}) \otimes_{U(\mathfrak{h} \otimes \mathbb{C}[t] \oplus \mathbb{C}c)} \mathbb{C} \cong S(\widehat{\mathfrak{h}}^-) \text{ (linearly),}$$

$\mathfrak{h} \otimes \mathbb{C}[t]$ acting trivially on \mathbb{C} and c acting as 1. Fock space for the lattice theory is

$$V_L = M(1) \otimes \mathbb{C}^\varepsilon[L] \cong S(\widehat{\mathfrak{h}}^-) \otimes \mathbb{C}[L] \text{ (linearly)}$$

with the usual tensor product grading. The Virasoro vector is $\frac{1}{2} \sum_i h_i(-1)h_i$, the sum ranging over *any* orthonormal basis $\{h_i\}$ of \mathfrak{h} .

For $\alpha \in \mathfrak{h}$ write $\alpha(n) := \alpha \otimes t^n$, $\alpha(z) := \sum_{n \in \mathbb{Z}} \alpha(n) z^{-n-1}$, $z^\alpha : e^\beta \mapsto z^{(\alpha, \beta)} e^\beta$, and set

$$Y(e^\alpha, z) := \exp \left(\sum_{m=1}^{\infty} \alpha(-m) \frac{z^m}{m} \right) \exp \left(- \sum_{m=1}^{\infty} \alpha(m) \frac{z^{-m}}{m} \right) e^\alpha z^\alpha,$$

and for $v = \alpha_1(-n_1) \dots \alpha_k(-n_k) \otimes e^\alpha \in V_L$ ($n_i \geq 1$) set

$$Y(v, z) := \left(\frac{1}{(n_1 - 1)!} \left(\frac{d}{dz} \right)^{n_1 - 1} \alpha_1(z) \right) \dots \left(\frac{1}{(n_k - 1)!} \left(\frac{d}{dz} \right)^{n_k - 1} \alpha_k(z) \right) Y(e^\alpha, z),$$

with the usual normal ordering conventions.

For $\gamma, \rho \in L$ we have

$$e^\gamma(n)e^\rho = \begin{cases} 0 & \text{if } (\gamma, \rho) \geq -n \\ \varepsilon(\gamma, \rho) e^{\gamma+\rho} & \text{if } (\gamma, \rho) = -n - 1 \\ \varepsilon(\gamma, \rho) \gamma(-1) e^{\gamma+\rho} & \text{if } (\gamma, \rho) = -n - 2 \\ \frac{1}{2} \varepsilon(\gamma, \rho) (\gamma(-2) e^{\gamma+\rho} + \gamma(-1)^2 e^{\gamma+\rho}) & \text{if } (\gamma, \rho) = -n - 3 \end{cases}$$

REFERENCES

- [BZHS] D. Ben-Zvi, R. Heluani and M. Szczesny, Supersymmetry of the chiral de Rham complex, *Compos. Math.* **144** (2008), 503–521.
- [dS] A. de Sole, Vertex algebras generated by primary fields of low conformal weight, MIT PhD thesis, 2003.
- [DM] C. Dong and G. Mason, Integrability of C_2 -Cofinite Vertex Operator Algebras, *IMRN*, Article ID 80468 (2006), 1–15.
- [EOT] T. Eguchi, H. Ooguri and Y. Tachikawa, Notes on the K3 Surface and the Mathieu group M_{24} , *Exp. Math.* **20** (2011), 91–96.
- [ET] T. Eguchi and A. Taormina, Unitary representations of the $N=4$ superconformal algebra, *Phys. Lett.* **B196** (1987), 75–81.
- [FLM] I. Frenkel, J. Lepowsky and A. Meurman, *Vertex Operator Algebras and the Monster*, Academic Press, New York, 1998.
- [K] V. Kac, *Vertex Algebras for Beginners*, AMS, Providence, RI., 1998.
- [MTY] G. Mason, M. Tuite and G. Yamskulna, Super conformal field theories defined by odd Niemeier lattices, in preparation.
- [S] B. Song, The global sections of the chiral de Rham complex on a Kummer surface, *IMRN* (2016), 4271–4296.

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF CALIFORNIA, SANTA CRUZ
E-mail address: gem@ucsc.edu

SCHOOL OF MATHEMATICS, STATISTICS AND APPLIED MATHEMATICS, NATIONAL UNIVERSITY OF IRELAND GALWAY, GALWAY, IRELAND
E-mail address: michael.tuite@nuigalway.ie

DEPARTMENT OF MATHEMATICS, ILLINOIS STATE UNIVERSITY, NORMAL, IL
E-mail address: gyamsku@ilstu.edu